EUROPEAN UNION AND THE SOLOW MODEL: Empirical Contribution to Theories of Economic Growth*

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Abstract

The following article deals with the basic Solow model of economic growth for the states of the EU-28 and empirically confirms its conclusions. Based on growth accounting, and by analyzing different determinants of the respective steady states of each country, we confirm the importance of capital accumulation. Moreover, the model confirms the existence of both, the absolute and conditional convergence, due to the relative homogeneity of the EU-28. Furthermore, the model developed implies values of the speed of convergence corresponding to the reality observed. Lastly, the model confirms that the Eastern Enlargement represented an acceleration in absolute convergence.

Keywords: Solow model; Steady state; European Union; Economic convergence

JEL Classification: O40, O41, O4

1. Introduction

Economic growth is a crucial aspect of modern economics and has been extensively studied and analyzed by economists. Robert (1956) introduced one of the most important models of long-term economic growth. He recommended using a neoclassical production function with capital, labor and labor augmenting technological progress. Solow's largest contribution is considered to be the so-called growth accounting. Due to the given shape of the production function and the way that technologies enter production, it is possible to decompose the determinants of growth, and to determine the influence of each individual production factors multiplied by their income shares. In the considered two sector economy, assuming an exogenous savings rate and population growth, it is possible to achieve a so-called steady state of income. Specifically, a state in which the income per "effective worker" is no longer growing. This steady state is, as shown below, determined by the rate of savings (investment rate) and the rate of population growth. Since population growth rates and savings rates vary from country to country, these countries in effect reach different steady states. Simply stated, Solow's theoretical model assumes that an exogenous increase in the savings rate leads to an equal increase in the investment rate, and thus to an increase in capital accumulation and total output. On the other hand, the higher the population growth, the lower the income per capita.

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Some economists extend Solow's model to include human capital in order to improve the model's results and eliminate any collinearity between savings rates, population growth rates, and capital accumulation. Therefore, we try to test the extended model in this article as well. However, as the results show, the model extended by human capital does not improve the ability to explain the different levels of income per worker.

In contrast to endogenous theories of long-term economic growth, the dynamics of the Solow model directly encourage the analysis of real convergence. For homogeneous entities such as the European Union, we can consider a single and common steady state into which all member economies are headed. In the period of dynamic stochastic models of short-term equilibrium and endogenous growth theories, this article aims to draw attention to the significance of the Solow model in determining different income levels of different countries, while confirming the existence and the acceleration of real convergence of European Union's then new member states after 2004. Moreover, the article's purpose is to confirm the plausibility and adequacy of a simplistic framework, such as Solow's model, when it comes to analyzing different levels of income per capita and differences in the speed of convergence in different periods and across different groups of member states. This article is also a result of the author's interest in the economic functioning of the European integration especially with respect to income inequality across member states, and how each state can maximize its potential gains from being part of this economic experiment.

The article is structured as follows. Section 2 relates the article to the existing literature, section 3 introduces the Basic Solow Model, section 4 is an empirical verification of growth accounting, section 5 empirically verifies the main results of the basic Solow model. In section 6 we extend the basic model to include human capital, and in section 7 we empirically verify the extended model. Section 8 focuses on convergence dynamics and their empirical verification. Lastly, section 9 concludes.

2. Related Literature

In this article we build on the model firstly developed by Solow (1956). After the model had been somehow abandoned in favor of endogenous growth models, Solow's model made a comeback. This revival can be attributed to the seminal paper by Mankiw, Romer and Weil (1992), in which they present and prove that despite its simplicity, Solow's basic model is able to describe and fit observed stylized empirical facts significantly well. Our article is related to the mentioned literature in that it follows in the steps of the methodology presented by Mankiw, Romer and Weil (1992), but incorporates human capital in a slightly different manner than the aforementioned authors. Moreover, in contrast to previous research, our main focus is targeted towards the convergence dynamics inside the European Union in particular years, from which we try to analyze the impact of the Eastern enlargement on the convergence of the analyzed member states.

A different perspective on economic growth is offered by endogenous growth theories, which argue that economic growth is driven by the accumulation of knowledge and the development of new technologies. These theories highlight the importance of innovation, research and development, as demonstrated by Romer (1986), Romer (1994) and Romer (1989). We incorporate this notion in the form of the level of human capital, as compiled by Feenstra et al. (2016). Related to endogenous growth theories is the literature relating education and labor productivity to income inequality and economic growth, as propagated by Lee and Hanol (2016) and Lee and Hanol (2018). Moreover, as demonstrated on the particular case of China, Lee (2017) and Barro (2016) emphasize the role of institutions and governance in shaping the conditions for growth by facilitating investment, trade, and the diffusion of knowledge. This is also in line with Robinson's and Acemoglu's emphasis on inclusive institutions, as a mean of harvesting the full potential of extensive economic growth and transiting into an inclusive form of economic growth, as propagated by Robinson, Acemoglu and Johnson (2005).

In addition to these broad perspectives, there is also a growing body of literature on the relationship between economic growth and various factors, such as inequality, education, trade, and the environment.

Lastly, we make use of the insights and build on the mathematical models developed by Barro and Sala-I-Martin (2004), Acemoglu (2009) and David Romer (2012).

3. The Basic Solow Model

Let us first look at the mathematical derivation of the model. In order to achieve a specific solution of endogenous variables, we use a Cobb-Douglas production function in the following form (extensive form of production function)[1]

$$Y(t) = K(t)^{\alpha}\left[A(t)L(t)\right]^{(1-\alpha)} \tag{1}$$

where $Y(t)$ denotes the total real product at time t, $K(t)$ denotes the capital stock at time$t$, $A(t)$ denotes total employment and $A(t)$ denotes the stock of technology, again at time t. As already mentioned, the Solow model assumes exogenous savings rates, population growth and technological progress. In continuous time, exogenous rates of population growth and technological progress can be expressed as $\frac{L'(t)}{L(t)} = n$ and $\frac{A'(t)}{A(t)} = g$[2].

We utilize an intensive form of the production function so that the total stock of a given variable is normalized per effective worker AL. Thus, the intense form of the Cobb-Douglas production function is[3]:

$$y(t) \ - y(t)^{a}, \tag{2}$$

where $y(t) = \frac{Y(t)}{A(t)L(t)}$ and $k(t) = \frac{K(t)}{A(t)L(t)}$. From the law of motion of capital, we know, that:

$$K'(t) = I(t) - \delta K(t), \tag{3}$$

1 In addition, we utilize the Solow model in continuous time, similar to Acemoglu (2009, pp. 55-78).

2 Explanation in Mathematical Appendix A.

3 Derivation provided in Mathematical Appendix A.

where $I(t)$ indicates gross investment at time t and δ indicates the rate of depreciation of physical capital. In the case of equality of total investments and the depreciation of physical capital, the capital stock remains unchanged and so net investment is zero. Assuming equality of savings and investment we get that:

$$K'(t) = sY(t) - \delta K(t), \tag{4}$$

where s represents a constant and exogenous savings rate. By expressing the above equation per an effective worker, we get[4]:

$$k'(t) = sf\left(k(t)\right) - \left(n + g + \delta\right)k(t) \,. \tag{5}$$

The steady state defined as a state in which the capital stock is constant. This means that $k'(t) = 0$. If the economy is in its steady state, it is possible to express from Eq. (5) the capital stock corresponding to the given steady state using exogenous parameters as follows:

$$k(t)^{*} = \left[\frac{s}{\left(n + g + \delta\right)}\right]^{\left(\frac{1}{1-\alpha}\right)} \tag{6}$$

where the asterisk indicates the quantity in a steady state. From the intense form of the production function in Eq. (2) we know that $y(t) = k(t)^{a}$, and so

$$y(t)^{*} = \left[\frac{s}{\left(n + g + \delta\right)}\right]^{\left(\frac{\alpha}{1-\alpha}\right)} \tag{7}$$

Through the previous steps we have only obtained quantities expressed per effective workers. Since we cannot observe effective workers in reality, for the purpose of empirical verification it is necessary to convert and express Eq. (7) per "ordinary" workers. This can be easily achieved by multiplying both sides of the equation by the expression $A(t)$, by which we obtain:

$$\frac{Y(t)^{*}}{L(t)} = A(t)\left[\frac{s}{\left(n + g + \delta\right)}\right]^{\left(\frac{\alpha}{1-\alpha}\right)} \tag{8}$$

Since $A(t) = A(0)e^{gt}$, by substituting this fact into Eq. (8) and taking logarithms of both sides of the equation we obtain:

$$\ln\frac{Y(t)}{L(t)} = \ln A(0) + gt + \frac{\alpha}{1-\alpha}\ln(s) - \frac{\alpha}{1-\alpha}\ln(n + g + \delta) \tag{9}$$

Before proceeding to the empirical verification of Eq. (9), let us first determine the effects of technological progress on the growth rates of European Union incomes using growth accounting techniques.

4 Proof can be found in Mathematical appendix A.

4. Empirical Verification of Growth Accounting

In order to decompose the growth rate of income to its individual factors, an overall differential of the production function with respect to time must be taken in the following manner[5]:

$$\frac{\dot{Y}}{Y} = \frac{F_A A}{Y}\frac{\dot{A}}{A} + \frac{F_K K}{Y}\frac{\dot{K}}{K} + \frac{F_L L}{Y}\frac{\dot{L}}{L} \tag{10}$$

Let us denote the growth rates of product, capital, and labour as $g_y = \frac{\dot{Y}}{Y}$, $g_K = \frac{\dot{K}}{K}$ and $g_L = \frac{\dot{L}}{L}$, respectively. Furthermore, we define $SR = \frac{F_A A}{Y}\frac{\dot{A}}{A}$. In a perfectly competitive environment such as Solow's economy, where $F_K = R$ and $F_L = W$, it is possible to rewrite the above Eq. (10) as:

$$g_y = SR + \alpha_K g_K + \left(1-\alpha_K\right) g_L \tag{11}$$

Or, by expressing for *SR* as:

$$SR = g_y - \alpha_K g_K - (1-\alpha_K) g_L \tag{12}$$

Because we have specific data on the growth rate of factors of production and their share of total income, it is not difficult to calculate the share of unexplained factors in relation to economic growth. In order to be able to estimate Eq. (12) empirically, it is necessary to modify it as follows:

$$SR_{jt,t+1} = g_{Y,jt,t+1} - \alpha_{K,jt,\,t+1} g_{K,jt,t+1} - \left(1-\alpha_{K,jt,t+1}\right) g_{L,jt,t+1} \tag{13}$$

The variable $\alpha_{K,j_{t,t+1}}$ does not indicate the growth rate of shares of capital incomes, but the average value of the share between times t and $t + 1$. In other words, that $\alpha_{K,j_{t,t+1}} = \frac{\alpha_{K,j_t} + \alpha_{K,j_{t+1}}}{2}$[6]. Because the expression $\alpha_{K,j_{t,t+1}}$ is more accurate the smaller the period between observations, Eq. (13) is tested for each period separately.

Eq. (13) was estimated for each year, i.e., from 1991 to 2017. Then the individual year-on-year changes of the analyzed variables were averaged over the period in every country[7]. In order to increase the level of aggregation, the national averages obtained over the last

5 We denote partial derivatives of the production function with respect to the individual factors of production as F_A , F_K a F_L.

6 Acemoglu (2009, p. 89) states that "the expression $\frac{\alpha_{K,j_t} + \alpha_{K,j_{t+1}}}{2}$ is a pretty good approximation to the expression $\alpha_{K,j_{t,t+1}}$, when the difference between t and $t + 1$ is small and the capital-labour ratio does not change much during this time interval".

7 In other words, the annual average growth rates of the variables in individual countries for the period 1991–2017.

twenty-seven years have been averaged over all European countries[8]. All data used are from *Penn 9.1 World Tables* (2015). The resulting equation estimated as described above can be written (in percentages) as:

$$\begin{array}{ccccccc} SR & = & g_y & - & \alpha_K g_K & - & (1-\alpha_K) g_L \\ 0.7\% & = & 3.1\% & - & 2.3\% & - & 0.1\% \\ [17.6\%] & & [100\%] & & [79.4\%] & & [3{,}0\%] \end{array} \tag{13}$$

For clarity, the share of the growth rate of the Solow Residual (and other variables) in the growth rate of economic output is given in square brackets. It is worth noting that the share of the average change in the capital stock is responsible for almost eighty percent of the average growth rate of total output. These results show that the growth of the capital stock is responsible for a large part of the growth of total output. Therefore, since we have found a significant positive effect of this capital accumulation on economic growth, we can explain the different levels of outputs (per worker) in individual countries of the European Union through the analysis of capital dynamics (using the savings rate and effective capital depreciation).

5. Empirical Verification of The Basic Solow Model

Specification of the model

In order to estimate Eq. (9) empirically, it is necessary to add some assumptions about technological progress. Firstly, the same growth rate of technological progress is assumed in all examined countries, and thus that expression g in Eq. (9) is not denoted by the j^{th} index. In addition, Acemoglu (2009, pp. 106-107) adds the assumption of orthogonal technology, which ensures the elimination of multicollinearity on the right-hand side of the estimated equation. The correlation of the explanatory variables would be evident in the case of significantly biased estimates of the elasticity of output per worker with respect to the savings rate and effective depreciation of capital, respectively. However, our estimation of the regression equation results in consistent and non-biased estimates of the parameters corresponding to observed stylized facts. If the assumptions mentioned above are met, it is possible to convert Eq. (9) to its estimated form:

$$\ln\left[\frac{Y}{L}\right]_j = constant + \frac{\alpha}{1-\alpha}\ln(s_{k,j}) - \frac{\alpha}{1-\alpha}\ln\left(n_j + g + \delta_{j,k}\right) + \varepsilon_j \,. \tag{14}$$

8 The result is therefore the (approximate) average annual growth rate of output, factors of production (multiplied by their share of total income) and the Solow residual for the member States of the European Union in 1991-2017.

Data

All data used are available from *Penn 9.1 World Tables* compiled by Feenstra, Inklaar, and Timmer (2015). Moreover, monetary variables are expressed in monetary units and converted at the purchasing power parity exchange rate. For the expression $\ln\frac{Y}{L}$ data on the real gross domestic product per worker is used – by worker we mean an economically active person. Both variables are from 2017 for all analyzed economies. Instead of the savings rate, the investment rate is used. That is because the investment rate is much more indicative of the degree of capital accumulation (hence the level of income) than the savings rate[9]. Investments are calculated as real domestic absorption less real domestic and government consumption. The investment rate is then expressed as the ratio of the calculated investment and output in each year and country. For the purpose of empirical verification, the average investment rate in the years 1990–2017 is used for individual states.

Based on the available data, the average growth rate of real output for the countries of the European Union in the years 1990-2017 was roughly 2.7 %. This value is therefore used as a common and constant growth rate *g*. The average growth rate of the labour force was calculated as a geometric mean over the observed period. The depreciation rate is the average depreciation rate in the years 1990-2017. The investment rates, population growth rates and capital depreciation rates are country specific.

On the following page, Eq. (15) is estimated for three different groups of countries - the EU28 (all European Union countries), the countries establishing the European Communities (from now on denoted as "the Six") and the new member states (which joined in 2004, from now on denoted as "NMS"). The regression equation therefore has the following form, where we assume that coefficients β_1 and β_2 have the same values but opposite signs. The results of the estimated equation can be found in Table 1.

$$\ln\frac{Y}{L}_j = constant + \beta_1 \ln\left(s_{k,j}\right) - \beta_2 \ln\left(n_j + g + \delta_{k,j}\right) + \varepsilon_j \qquad (15)$$

Results

Three aspects of the displayed results support the Solow model in the case of the EU28 countries. Individual coefficients indicate the elasticity of income with respect to given variables. Both coefficients have almost the same values with opposite signs. Moreover, both coefficients and the model as a whole are statistically significant. In the case of applying Eq. (15) to the founding states, we see that the coefficient of the savings rate is not statistically significant. On the other hand, the coefficient of effective capital depreciation is statistically significant. Although the model is still statistically significant, due to the inclusion of an "unnecessary" investment rate it is significant only at the 5% level of significance. Finally, when applying Eq. (15) to NMS, we see that the size of the coefficients (in absolute value) does not match. However, unlike when tested for "the Six", all coefficients are statistically significant. The model as a whole is significant even at the one percent level of significance.

9 Especially with regard to the openness and high capital mobility of the analysed economies.

Based on the index of determination, we can state that the model is able to explain up to 70 % of the variability of GDP per worker in the case of EU28, 91 % for the Six and 82 % when testing the model on NMS data, only on the basis of the variability in investment rates, population growth and capital depreciation across countries. When restricting the coefficients, the index of determination is even higher – roughly 0.71, 0.93 a 0.76 respectively. The high indexes of determination confirm the fact that, in contrast to the model's conclusions about the high impact of "technological progress" on the levels of income, investment and population growth rates' variability allow us to explain the vast majority of the variability in the level of income per worker.

Table 1: Estimation of Eq. (15) for EU28, the Six and NMS

Dependent variable: log GDP per worker in 2017

Group of countries:	***EU28***	***The Six***	***NMS***
Constant	14,77*** (0.49)	14.25*** (0.40)	13.34*** (0.34)
ln(*s*)	0.84*** (0.18)	0.62 (0.32)	0.28* (0.13)
ln(*n* + *g* + *δ*)	−0.83*** (0.19)	−0.67** (0.15)	−0.65*** (0.12)
F − stat	32.35***	25.25**	21.17***
R^2_{adj}	0.70	0.91	0.82
Restricted regression:			
Constant	14.77*** (0.43)	14.26*** (0.33)	13.14*** (0.36)
ln(*s*) − ln(*n* + *g* + *δ*)	0.84*** (0.1)	0.66*** (0.08)	0.47*** (0.09)
F − stat	67.29***	67.05***	21.17***
R^2_{adj}	0.71	0.93	0.76
Implied a	0.46	0.4	0.32

Table 1: Estimation of equation (15) for EU28, the Six and NMS

Source: Based on own calculations. Data obtained from Feenstra, Inklaar and Timmer *(2015)*. For GDP per worker data on the real gross domestic product per worker is used. The savings rate - *s* - is expressed as the average ratio of investment and output. The average growth rate of real output - *g* - for the European Union was approximately 2.7 %. The average growth rate of the labor force - *n* - is calculated as a geometric mean over the observed period. The depreciation rate - *δ* - is the average depreciation rate of capital. GDP per worker, *s*, *n* and *δ* are all country specific. All variables are averaged throughout 1990-2017.

(***) 1 % statistical significance, (**) 5 % and (*) 10 %.

Standard errors in parentheses under values of coefficients.

Implied α (capital's share of income) after restricting the coefficients is in the case of EU28 around 0.46, in the case of the Six 0.4, and in the case of NMS only 0.321. Upon closer analysis of EU28 data, we find that the average value of α was in fact in the period 1990-2017 equal to 0.43, which is almost identical to the value implied by our estimate. Thus, the model seems to approximate the observed reality very well for European countries in the last thirty years. In addition, this means that it is not even necessary to reduce the value α by re-evaluating our concept of capital and adding human capital as an explanatory variable into the estimated equation. This means that we can already assume the relative insignificance of different levels of human capital in different countries in explaining the different levels of output per worker.[10]

Looking at Table 1, we can see that Eq. (15) has achieved the best results for the founding states of the EU. We can thus assume that these countries have already reached their specific steady state. It is therefore important to note that the division of states into "east and west" helps us clarify the fact that individual groups of states are most likely situated at different distances from their respective steady states (judging by the fact that the estimates of the Six have a much higher R^2_{adj} than NMS). Despite the satisfactory results of the model, let us now try to improve the estimation by incorporating human capital into the analysis.

6. Extending Solow Model by Human Capital

Again, we utilize the neoclassical Cobb-Douglas production function, into which, in contrast to Eq. (1), we incorporate human capital as follows:

$$Y(t) = K(t)^{\alpha} H(t)^{\beta} \left[A(t) L(t) \right]^{(1-\alpha-\beta)}, \tag{16}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, a $\alpha + \beta < 1$. Under these assumptions, the above equation can be converted to an intensive form in the following manner[11]:

$$y(t) = k(t)^{\alpha} h(t)^{\beta}. \tag{17}$$

Assuming that human capital "behaves" in the same way as human capital, i.e., it is accumulated on the basis of exogenous rates of investment into human capital, and also depreciates in the same fashion, the levels of both capitals can be expressed in the steady state as follows:

$$k^* = \left\{ \left[\frac{s_k}{(n+g+\delta_k)} \right]^{(1-\beta)} \left[\frac{s_h}{(n+g+\delta_h)} \right]^{\beta} \right\}^{\left(\frac{1}{1-\alpha-\beta}\right)}$$

$$h^* = \left\{ \left[\frac{s_k}{(n+g+\delta_k)} \right]^{\alpha} \left[\frac{s_h}{(n+g+\delta_h)} \right]^{(1-\alpha)} \right\}^{\left(\frac{1}{1-\alpha-\beta}\right)} \tag{18}$$

10 Because physical capital and its effective depreciation explain the vast majority of variability, and because it is clear that there are other factors that can explain to us much better the observed variability of data (e.g., the existence of a single currency).

11 As in the case of the basic Solow model.

Substituting these facts into Eq. (17) we obtain:

$$y(t)^* = \left[\frac{s_k}{\left(n+g+\delta_k\right)}\right]^{\frac{\beta}{1-\alpha-\beta}} \left[\frac{s_h}{\left(n+g+\delta_h\right)}\right]^{\frac{\alpha}{1-\alpha-\beta}}. \tag{19}$$

Since again we do not know effective workers, we will apply the same assumptions here as in the case of deriving the output per worker in the steady state in the basic model. So, in a similar way we see that[12]:

$$\frac{Y(t)^*}{L(t)} = A(t)\left[\frac{s_k}{\left(n+g+\delta_k\right)}\right]^{\left(\frac{\alpha}{1-\alpha}\right)} h^{*\left(\frac{\beta}{1-\alpha}\right)}. \tag{20}$$

Substituting for the technology and taking logarithms of both sides of the equation we get the shape:

$$\ln\frac{Y(t)}{L(t)} = \ln A(0) + gt + \frac{\alpha}{1-\alpha}\ln(s_k) - \frac{\alpha}{1-\alpha}\ln\left(n+g+\delta_k\right) + \frac{\beta}{1-\alpha}\ln(h^*) \tag{21}$$

7. Empirical Verification of the Extended Solow Model

Specification of the model

The assumptions given in the section concerning the empirical verification of the basic Solow model also apply to the extended model. The estimated equation, where the coefficients indicate the individual elasticities of the income per capita with respect to the explanatory variables, is:

$$\ln\frac{Y}{L_j} = const + \frac{\alpha}{1-\alpha}\ln(s_{k,j}) - \frac{\alpha}{1-\alpha}\ln\left(n_j + g + \delta_{k,j}\right) + \frac{\beta}{1-\alpha}\ln(h^*{}_j) + \varepsilon_j . \tag{22}$$

12 Derivation of Eq. (20) can be found in Mathematical Appendix

Table 2: Estimation of Eq. (23) for EU28, the Six and NMS

Dependent variable: log GDP per worker in 2017

Group of countries:	***EU28***	***The Six***	***NMS***
Constant	14.20*** (0.55)	14.20*** (0.50)	13.53*** (0.38)
ln(s_k)	0.87*** (0.17)	0.71 (0.48)	0.30* (0.13)
ln(h^*)	0.55* (0.29)	0.12 (0.39)	−0.26 (0.24)
ln($n + g + \delta$)	−0.84*** (0.19)	−0.66* (0.17)	−0.60*** (0.18)
F – stat	25.08***	11.80*	14.97***
R^2_{adj}	0.73	0.87	0.82
Restricted Rregression:			
Constant	14.22*** (0.45)	14.20*** (0.41)	13.45*** (0.40)
ln(s) – ln($n + g + \delta$)	0.85*** (0.10)	0.67*** (0.10)	0.45*** (0.08)
ln(h^*)	0.55* (0.28)	0.10 (0.26)	−0.36 (0.25)
F – stat	39.17***	36.40**	18.30**
R^2_{adj}	0.74	0.91	0.79
Implied α	0.46	0.40	0.31
Implied β	0.30	0.06	−0.25

Table 2: Estimation of equation (15) for EU28, the Six and NMS.

Source: Based on own calculations. Data obtained from Feenstra, Inklaar and Timmer *(2015)*. For GDP per worker data on the real gross domestic product per worker is used. The savings rate - s - is expressed as the average ratio of investment and output. The average growth rate of real output - g - for the European Union was approximately 2.7 %. The average growth rate of the labour force - n - is calculated as a geometric mean over the observed period. The depreciation rate - δ - is the average depreciation rate of capital. The human capital proxy - h^* - is the average human capital index. GDP per worker, s, n, δ and h^* are all country specific. All variables are averaged throughout 1990-2017.

(***)1% statistical significance, (**) 5 % and (*) 10 %.

Standard errors in parentheses under values of coefficients.

Data

Since most variables and data were described in the previous section, we will focus here only on the expression h^*_j. World-renowned authors Barro and Lee (2013) and Lee and Lee (2016) are among the most cited authors in examining the impact of human capital on long-term economic growth. Therefore, the authors of this work primarily tried to first use the statistics on school attendance from Barro's and Lee's tables as a proxy variable (where we can find specific percentages of school attendance for specific age categories and genders)[13].

Nevertheless, the authors of this article make use of the so-called "human capital index", which partially builds upon the knowledge of Barro and Lee and is based on their logic, but also adds the facts set by critics of these authors and their methodology. Alternatively, for some countries, the human capital index uses different methodologies for measuring human capital other than Barro and Lee, specifically, the methodology of Cohen and Soto (2007). The human capital index is normalized across all countries, and therefore its values can be used as a proxy variable expressing the *level* of human capital in the economy.[14]

The regression equation thus takes the following shape:

$$\ln \frac{Y}{L}_j = constant + \beta_1 \ln s_{k,j} - \beta_2 \ln\left(n_j + g + \delta_{k,j}\right) + \beta_3 \ln\left(h^*_{\ j}\right) + \varepsilon_j \tag{23}$$

Results

The results in Table 2 show that the estimates were again made for three groups of countries, as in the case of the previous estimate. It is evident that by incorporating human capital into the basic model, we have achieved somewhat insignificantly better results only in the case of the EU28 countries. According to our model, human capital is a statistically insignificant variable for other groups of countries. However, based on growth accounting and the empirical verification of the model without human capital, we have already assumed that human capital would not play too much of a role. The value of the parameter α implied by the basic model was almost identical to the value actually observed in the real data.

Furthermore, it should also be noted that although the involvement of human capital does not change the values of the investment rate coefficients and the effective depreciation of physical capital, whether in terms of size or sign, there is a partial reassessment of income shares of the respective factors of production, where now only 0.14 of total incomes falls on "raw" labour, 0.3 of total income flows to human capital, which is due to our specification also considered a form of labour, but in a relatively different light, and 0.46 on physical capital –incomes flowing to capital owners. Capital (physical and human) therefore has a much larger role in this model than in the previous model (in terms of the share of total income). It is noteworthy, however, that the extended model can, only on the basis of the investment rate, effective depreciation, and a proxy for human capital, clarify roughly 75 % of the variability of incomes per worker in the European Union in the years 1990-2017.

13 A methodology like Mankiw, Romer and Weil (1992) since the authors make usage of the rate of investment into human capital, and not the level of investment. Thus, our analysis differs in this aspect.

14 We refer the interested reader about the method of compilation of the human capital index to Feenstra, Inklaar and Timmer (2016).

8. Absolute and Conditional Real Convergence in the Solow Model

The previous two sections assumed economies in their steady states. However, most economies have not yet reached their steady state. Therefore, let us move on to the analysis of another important area that the Solow model deals with, and that is the analysis of convergence relations and their dynamics.

The advantage of Solow's model lies, among other things, in its ability to elucidate the causes of economic convergence. Theories of endogenous growth usually do not predict any economic convergence, rather the contrary. They imply that despite the same preferences and technologies, output levels may differ from country to country. Most models of endogenous growth assume constant or even increasing returns from the variable factor of capital, which results in a different shape of the production function. In these models, there is usually no permanent state to which the economy converges.

Proponents of endogenous growth theories often reject the Solow model based on the claim that there is no convergence in real data. However, for the most part, the authors' critiques are related to absolute convergence. This is in line with our understanding of the Solow model, since we are not of the view that the Solow model implies absolute convergence. The Solow model assumes that each economy heads towards its own steady state over time, and not that all economies are heading for the same common steady state. However, it is worth noting, that in the case of homogenous groups of countries – the states of the European Union – it is possible to partially abstract and try to even test for the existence of absolute convergence. The reason is that since the European Union is a relatively homogeneous grouping, we can try to assume a common steady state for all Member States. We undertake this step mainly in order to find out and confirm whether the year 2004 represented an acceleration of the real convergence of the analysed economies. By solving a set of differential equations (the mathematical derivation can be found in Appendix A) we reach the necessary form of the equation to be estimated, which is that:

$$y(t) \simeq y^{*} + e^{-\lambda t}\left[y(0) - y^{*}\right]. \tag{24}$$

Empirical verification of absolute convergence

The concept of absolute convergence says that the lower the economy starts, the faster it grows. Mathematically, this can be illustrated as follows:

$$g_{Y\,j,t,t-1} = \beta_0 + \beta_1 \ln(Y_{j,t-1}) + \varepsilon_{j,t}, \tag{25}$$

where $g_{Y\,j,t,t-1}$ indicates the growth rate of the output of the *j*-th country between *t* and $t-1$ and $y_{j,t-1}$ indicates the output of the *j*-th country at time *t*. For absolute convergence to be present, we require that coefficient β_1 be negative and statistically significant. In order to use this form in accordance with the previous derivation, it is necessary to start from Eq. (24). Subtracting $y(0)$ from both sides of the equation yields:

$$y(t)-y(0) \simeq e^{-\lambda t}y(0)-y(0)+y^{*}-e^{-\lambda t}y^{*}$$

$$y(t)-y(0) \simeq y(0)(e^{-\lambda t}-1)+(1-e^{-\lambda t})y^{*} \tag{26}$$

$$y(t)-y(0) \simeq (1-e^{-\lambda t})y^{*}-(1-e^{-\lambda t})y(0)$$

The above equation must also apply to the natural logarithms of the given variables, so that:

$$\ln\left(y(t)\right)-\ln\left(y(0)\right) \simeq \left(1-e^{-\lambda t}\right)\ln\left(y^{*}\right)-\left(1-e^{-\lambda t}\right)\ln\left(y(0)\right). \tag{27}$$

On the left-hand side of the equation, we see the difference of the logarithms of the outputs at times t and 0. This value is approximately equal to the average growth rate of the product. If, instead of the first term of the right-hand side of the equation – the steady-state output – we substitute its parametric expression from Eq. (7), we get:

$$\begin{aligned}\ln\left(y(t)\right)_j-\ln\left(y(0)\right)_j \\ &= const+\left(1-e^{-\lambda t}\right)\frac{\alpha}{1-\alpha}\ln(s_{k,j})-\left(1-e^{-\lambda t}\right)\frac{\alpha}{1-\alpha}\ln\left(n_j+g+\delta_{k,j}\right) \\ &-\left(1-e^{-\lambda t}\right)\ln\left(y(0)\right)+\varepsilon_j\ .\end{aligned} \tag{28}$$

To test for absolute convergence, it is sufficient to omit the variables expressing the steady state of the economy. Keeping these explanatory variables in the equation would signify testing for conditional convergence. In this section, we will estimate three equations. Specifically, for three different periods – 1990-2017, 1990-2003 and 2004-2017. The reason for dividing the whole period into two is to detect the change in the rate of convergence between the individual periods. The estimated equations therefore have the forms:

$$\ln\left(y(2017)\right)_j-\ln\left(y(1990)\right)_j = const-(1-e^{-\lambda t})\ln\left(y(1990)\right)+\varepsilon_j \tag{29a}$$

$$\ln\left(y(2003)\right)_j-\ln\left(y(1990)\right)_j = const-\left(1-e^{-\lambda t}\right)\ln\left(y(1990)\right)+\varepsilon_j \tag{29b}$$

$$\ln\left(y(2017)\right)_j-\ln\left(y(2004)\right)_j = const-(1-e^{-\lambda t})\ln\left(y(2004)\right)+\varepsilon_j \tag{29c}$$

Results

As can be seen in Table 3, the coefficients are negative, as was required. Except for the states of the Six, all coefficients and models as a whole are statistically significant. Furthermore, the variability in average income per worker growth rates for the EU28 and NMS is explained fairly well by the initial value of income per worker. The implied rate of convergence is highest for the New Member States, which is also quite logical, since in 1990 these countries were much lower in terms of the level of income per worker than other countries. Thus, the "catching-up" effect is evident, where poorer states catch-up with richer states. Another interesting fact is the implied lambda for the EU28 countries, since in the previous section we had already mentioned that the average growth rate of European Union output in the period 1990-2017 has been about 3%. This generally corresponds to the results shown above. The low speed of convergence for the countries of the Six signifies their relative proximity to their steady state, where convergence dynamics do not play an important role as for the New Member States, which are in turn (based on the output of the estimation) relatively far from their steady state. For the Six, however, the coefficients and the model are insignificant.

Table 3: Estimation of Eq. (29a) for EU28, the Six and NMS

Dependent variable: the difference between log GDP per worker in 2017 and 1990

Group of countries:	*EU28*	*The Six*	*NMS*
Constant	6.40*** (0.82)	3.20 (1.53)	8.54*** (1.34)
$\ln\left(\frac{Y}{L_{90}}\right)$	−0.54*** (0.08)	−0.26 (0.14)	−0.75*** (0.13)
F – stat	48.11***	3.14	33.44***
R^2_{adj}	0.64	0.30	0.78
Implied λ	0.03	0.01	0.05

Table 3: Estimation of Eq. (29a) for EU28, the Six and NMS

Source: Based on own calculations. Data obtained from Feenstra, Inklaar and Timmer *(2015)*.

For GDP per worker data on the real gross domestic product per worker are used in the years 2017 and 1990, respectively.

*** 1 % statistical significance, ** 5 % a * 10 % .

Standard errors in parentheses under values of coefficients.

The results in Tables 4 and 5 below are very interesting from several points of view. It is advisable to compare the outputs of both tables. Let us focus primarily on the EU28. Here we confirm our hypothesis about the acceleration of absolute convergence (based on the implied value of λ). Also noted should be the value of the output coefficient, that actually decreased in the latter period. Last but not least, it should be noted that the initial output can much better explain the variability of growth rates in the second period, compared to the first period (based on adjusted indexes of determination).

Table 4: Estimation of Eq. (29b) for EU28, the Six and NMS

Dependent variable: the difference between log GDP per worker in 2003 and 1990

Group of countries:	***EU28***	***The Six***	***NMS***
Constant	2.75*** (0.80)	3.02* (1.21)	6.13** (2.42)
$\ln\left(\frac{Y}{L_{90}}\right)$	−0.23*** (0.07)	−0.25* (0.11)	−0.56** (0.23)
F – stat	9.11***	5.30*	5.69*
R^2_{adj}	0.23	0.46	0.34
Implied λ	0.02	0.02	0.06

Table 4: Estimation of Eq. (29b) for EU28, the Six and NMS
Source: Based on own calculations. Data obtained from Feenstra, Inklaar and Timmer *(2015)*.
For GDP per worker data on the real gross domestic product per worker are used in the years 2003 and 1990, respectively.

*** 1 % statistical significance, ** 5 % a * 10 %.

Standard errors in parentheses under values of coefficients.

As for the countries of the Six, we see that while in the first period there is a statistically significant convergence, in the second period we observe a relative absolute divergence, based on a statistically insignificant coefficient and model. Thus, the global and consequently the European debt crisis seems to have caused the countries of the Six to diverge from their unified and common steady state. Again, however, it should be noted that the coefficient is statistically insignificant in the second period, compared to the first period, so the level of output per worker in 2004 did not affect the subsequent growth rates of output per worker for the countries of the Six. For the New Member States, the coefficients and models are statistically significant in both periods, but we see a slowdown in the pace of convergence in the second period. This result signifies that the New Member States are already closer to their common steady state and thus the speed of convergence slowed down. Lastly, the importance of the output levels in 2004 should be noted in clarifying the subsequent growth rates of outputs – output in 2004 can explain up to 80% of the different growth rates of outputs of the New Member States in 2004-2017. The lower value of the adjusted index of determination in the first period is due to the relatively turbulent economic environment of the New Member States in the 1990s.

Table 5: Estimation of Eq. (29c) for EU28, the Six and NMS

Dependent variable: the difference between log GDP per worker in 2017 and 2004

Group of countries:	*EU28*	*The Six*	*NMS*
Constant	4.48*** (0.71)	0.06 (0.99)	5.75*** (0.9)
$\ln\left(\frac{Y}{L_{04}}\right)$	−0.38*** (0.06)	0.02* (0.09)	−0.50** (0.08)
F – stat	33.91***	0.04	34.36***
R^2_{adj}	0.55	−0.25	0.79
Implied λ	0.03	−0.001	0.05

Table 5: Estimation of Eq. (29c) for EU28, the Six and NMS

Source: Based on own calculations. Data obtained from Feenstra, Inklaar and Timmer *(2015)*.
For GDP per worker data on the real gross domestic product per worker are used in the years 2017 and 2004, respectively.

*** 1 % statistical significance, ** 5 % a * 10 % .

Standard errors in parentheses under values of coefficients.

Empirical Verification of Conditional Convergence

We have already touched on the concept of conditional convergence. In the first parts of this article, we focused on the analysis of the steady states of individual countries under various groupings (either with or without the inclusion of human capital). In the previous subsection, we focused on absolute convergence, where we tried to analyze the dynamics of countries on their road to a common steady state. We will now combine the two parts and try to analyze the path of each economy to its specific steady state. In other words, we will try to find out whether, in fact, the further the economy is from its specific steady state, the faster it then grows. Remember that the equation characterizing conditional convergence is:

$$\begin{aligned}\ln\left(y(t)\right)_j - \ln\left(y(0)\right)_j = \\ = const + (1-e^{-\lambda t})\frac{\alpha}{1-\alpha}\ln(s_{k,j}) \\ -\left(1-e^{-\lambda t}\right)\frac{\alpha}{1-\alpha}\ln\left(n_j+g+\delta_{k,j}\right)-\left(1-e^{-\lambda t}\right)\ln\left(y(0)\right)+\varepsilon_j\end{aligned} \tag{30}$$

Table 6: Estimation of Eq. (30) for EU28 and NMS

Dependent variable: the difference between log GDP per worker in 2017 and 1990

Group of countries	*EU*28	NMS
Constant	12.10*** (1.51)	11.56*** (0.68)
$\ln\left(\frac{Y}{L_{90}}\right)$	−0.83*** (0.09)	−0.85*** (0.05)
$\ln(s_k)$	0.61*** (0.21)	0.14 (0.10)
$\ln(n+g+\delta)$	−0.65*** (0.21)	−0.63*** (0.09)
F – stat	31.43***	112.73***
R^2_{adj}	0.77	0.97
Implied λ	0.06	0.07
Restricted Rregression:		
Constant	12.09*** (1.48)	11.92*** (1.04)
$\ln\left(\frac{Y}{L_{90}}\right)$	−0.83*** (0.09)	−0.90*** (0.08)
$\ln(s) - \ln(n+g+\delta)$	0.63*** (0.15)	0.42*** (0.09)
R^2_{adj}	0.78	0.94
F – stat	49.05***	68.13***
Implied *α*	0.43	0.32
Implied *λ*	0.07	0.09

Table 6: Estimation of Eq. (30) for EU28 and NMS

Source: Based on own calculations. Data obtained from Feenstra, Inklaar and Timmer *(2015)*.

For GDP per worker data on the real gross domestic product per worker is used. The savings rate - s - is expressed as the average ratio of investment and output. The average growth rate of real output - *g* - for the European Union was approximately 2.7. The average growth rate of the labour force - *n* - is calculated as a geometric mean over the observed period. The depreciation rate - *δ* - is the average depreciation rate of capital. GDP per worker, *s, n* and *δ* are all country specific. All variables are averaged throughout 1990-2017 except for GDP per worker, for which data from 1990 and 2017 is used, respectively.

*** 1 % statistical significance, ** 5 % and * 10 %. Standard errors in parentheses under values of coefficients.

If we omit the constant, then the first two members of the right-hand side of the equation determine the steady state of a particular economy. The last term then only shows where the economy starts off. Thus, it is a combination of both factors affecting the individual growth rates of the economies analyzed in the previous sections. When estimating Eq. (30) we assume individual steady states of individual economies. We try to control for growth caused by a higher steady state of a particular economy. We want to prove that the higher growth rate does not have to be determined only by the initial level of output, but also by the level to which the economy is heading.

The estimates of Eq. (30) in this section are examined only for two groups of states in the years 1990–2017. Only the EU28 and NMS countries were chosen, as we have already found out (in Table 3) that there is no real convergence in the case of the Six[15]. Moreover, we neglect human capital since we have already confirmed its relative insignificance in determining the specific steady states of individual economies.

From Table 6 the incorporation of state-specific steady state determinants caused an increase in the speed of real convergence. We see that the index of determination reaches excellent values for both groups – 97 % of the average growth rate variability explained by our model in the case of the NMS is a striking result.

This means that ninety seven percent of the average growth rate variability of the NMS output was determined only by the distance of a particular economy from its specific steady state. In the case of the NMS, it is evident that the average level of investment (in physical capital) did not have a statistically significant impact on the average growth rate of output per worker in 1990–2017. However, in the case of the restricted regression, it reaches statistically significant coefficients in all cases. In addition, the values of the share of capital income implied by the model corresponds exactly to the values actually observed for both of the analyzed groups.

Conclusion

Although the Solow model is only a basic model of long-term economic growth, in the case of the European Union, the model was able to explain different levels of income only on the basis of investment rates and effective capital depreciation. Extending the model to include human capital did not improve the model's ability to clarify these differences. On the contrary, the basic textbook model implies values of the factors' shares of income corresponding to the actually observed shares in the data. For the founding states of the European Communities, the restricted model (in Table 1) was able to clarify up to 93% of the variability of income levels per worker, from which we deduced that the economies that started the European Communities have already reached their steady states.

Unlike endogenous growth models, one of the most important implications of the Solow model is the existence of real conditional convergence. Conditional with respect to a given specific set of parameters of an economy defining its particular steady state. In the last part of this article, we first combined both concepts of convergence for the states of the European Union and thus proved and confirmed the increase in the speed of convergence after 2004 (assuming all countries share the same common steady state). For the founding states, convergence took place only in the years 1990-2003. For the New Member States, convergence took place in both periods tested, but nonetheless at a slower pace in the second period. However, for all the Member States of the European Union, convergence accelerated in the second period, confirming the phenomenon of poorer economies catching up with the richer ones after their accession into the integration process. Lastly, the existence of conditional convergence was confirmed. For the countries of the European Union and New Member States we achieved values of capital's share of income that exactly correspond to the observed data. In addition,

15 Especially since, from our point of view, these economies are already in their steady state.

the model clarified nearly 80% of the variability in the real product growth rates of individual European economies.

* * *

Despite the achieved results, we do not reject the contributions of endogenous theories, but rather the contrary. We support them, because it is necessary to understand the determinants of exogenous variables entering the Solow model. We only point out that on the basis of different investment rates, effective physical capital depreciation, and initial outputs, it is possible to clarify the majority of the observed variability in both output levels and their growth rates.

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MATHEMATICAL APPENDIX

The Basic Solow Model

Population growth and technological progress in continuous time

As stated, in continuous time, exogenous rates of population growth and technological progress can be expressed as $\frac{L'(t)}{L(t)} = n$ and $\frac{A'(t)}{A(t)} = g$. These results are achieved under the assumption of exponential growth of the workforce and technological progress. And so that $L(t) = L(0)e^{nt}$ and $A(t) = A(0)e^{gt}$. By taking logarithms of both sides of the equations, we obtain $\ln L(t) = \ln L(0) + nt$ and $\ln A(t) = \ln A(0) + gt$. Derivating both sides of the equations with respect to time we get that $\frac{d \ln L(t)}{dt} = \frac{1}{L(t)} L'(t) = n$ and that $\frac{d \ln A(t)}{dt} = \frac{1}{A(t)} A'(t) = g$

Derivation of the intensive form of the production function

Assuming constant returns to scale, it is possible to multiply the extensive form of the production function by the expression $\frac{1}{A(t)L(t)}$, from which we obtain:

$$\frac{Y(t)}{A(t)L(t)} = \frac{\left\{K(t)^{\alpha}\left[A(t)L(t)\right]^{(1-\alpha)}\right\}}{A(t)L(t)} = K(t)^{\alpha}\left[A(t)L(t)\right]^{(-\alpha)} = \left\{\frac{K(t)}{\left[A(t)L(t)\right]}\right\}^{(\alpha)}$$

Derivation of the law of motion of capital for the intensive form

Below we provide a proof for the derivation of Eq. (5) from Eq. (4).

$$k'(t) = \frac{dk(t)}{dt} = \frac{d\left(\frac{K(t)}{A(t)L(t)}\right)}{dt} = \frac{K^{\cdot}(t)\left[A(t)L(t)\right] - K(t)\left[A(t)^{\cdot}L(t)\right]}{\left[A(t)L(t)\right]^2} = \frac{\dot{K}(t)\left[A(t)L(t)\right]}{\left[A(t)L(t)\right]^2} - \frac{K(t)\left[A(t)^{\cdot}L(t)\right]}{\left[A(t)L(t)\right]^2}$$

$$=\frac{K'(t)}{A(t)L(t)}-\frac{K(t)[A'(t)L(t)+A(t)L'(t)]}{A(t)L(t)A(t)L(t)}$$

$$=\frac{sY(t)}{A(t)L(t)}-\frac{\delta K(t)}{A(t)L(t)}-\frac{K(t)}{A(t)L(t)}\left(\frac{A'(t)L(t)}{A(t)L(t)}+\frac{A(t)L'(t)}{A(t)L(t)}\right)$$

$$sy(t)-\delta k(t)-k(t)\,(g+n)$$

which is equal to Eq. (5).

The Solow Model with Human Capital

Derivation of Eq. (20)

Due to the fact that we have data expressing the level of human capital rather than the level of investment in human capital (as opposed to physical capital), a slightly different derivation has to be used. So let us start from the equation determining the steady state amount of physical capital, in which net investment is zero. This means that:

$$s_k y^* = (n+g+\delta)k^*$$

$$s_k k^{*\alpha} h^{*\beta} = (n+g+\delta)k^*$$

$$k^* = \left[\frac{s_k h^{*\beta}}{(n+g+\delta_k)}\right]^{\left(\frac{1}{1-\alpha}\right)}$$

$$k^* = \left[\frac{s_k}{(n+g+\delta_k)}\right]^{\left(\frac{1}{1-\alpha}\right)} h^{*\left(\frac{\beta}{1-\alpha}\right)}$$

And since

$$y = k^{*\alpha} h^{*\beta},$$

then

$$y^* = \left[\frac{s_k}{(n+g+\delta_k)}\right]^{\left(\frac{\alpha}{1-\alpha}\right)} h^{*\left(\frac{\beta\alpha}{1-\alpha}\right)} h^{*\beta}$$

$$y^* = \left[\frac{s_k}{(n+g+\delta_k)}\right]^{\left(\frac{\alpha}{1-\alpha}\right)} h^{*\left(\frac{\beta\alpha+\beta(1-\alpha)}{1-\alpha}\right)}$$

$$y^* = \left[\frac{s_k}{(n+g+\delta_k)} \right]^{\left(\frac{\alpha}{1-\alpha}\right)} h^{*\left(\frac{\beta}{1-\alpha}\right)}$$

It is thus evident that both forms of capital should contribute to higher output. Multiplying both sides of the equation by $A(t)$ gives us Eq. (20).

Absolute and Conditional Real Convergence

Derivation of Eq. (24)

First of all, we perform a linearization of the net investment function around a steady state, in other words, we approximate the steady state to a linear function using Taylor expansion in the following manner[16]:

$$\dot{k} \simeq \left[\left. \frac{\partial \dot{k}(k)}{\partial k} \right|_{k=k^*} \right] \left(k - k^*\right),$$

The smaller the distance between k a k^*, the more accurate the approximation. If we define the expression $-\left.\frac{\partial \dot{k}(k)}{\partial k}\right|_{k=k^*}$ as λ, we can rewrite the above equation as:

$$\dot{k}(t) \simeq -\lambda\left(k(t) - k^*\right).$$

The above approximation tells us that if we are below steady state, net investment is positive. Conversely, if we are above steady state, net investment is negative. This equation also implies to us the fact that if we are out of the steady state, k is approaching k^* at a speed which is approximately proportional to its distance from k^*. This means that the growth rate of the expression $k(t) - k^*$ is approximately constant and is equal to $-\lambda$. By solving the above differential equation (26) we obtain:

$$k(t) \simeq k^* + e^{-\lambda t}\left[k(0) - k^*\right] \cdot$$

In addition, Romer (2012, p. 27) confirms that y is approaching y^* at the same pace at which k is approaching k^*, and thus:

$$y(t) \simeq y^* + e^{-\lambda t}\left[y(0) - y^*\right].$$

16 For more information about Taylor expansion and finding the particular solution of linear differential equations, we direct the reader to Chiang a Wainwright (2004, Ch.9 and Ch. 15)